Directions in the study of the double-critical graph conjecture

American University of Beirut
CMPS 396AC – Algorithmic Graph Theory

Jad Issa
jti01@mail.aub.edu

American University of Beirut, Computer Science

Charbel Abi Daher
charbel.abidaher@ul.edu.lb

Lebanese University, Electrical Engineering

December 2020

Abstract

A connected k-chromatic graph such that the deletion of any pair of adjacent vertices causes the remainder of the graph to have chromatic number $k-2$. Erdős conjectured in 1967 that K_k is the only k-chromatic double-critical graph. A solution of the double-critical conjecture goes a long way in solving the Erdős-Lovász Tihany conjecture. In this paper, we summarize the current directions for and hurdles against the current research in pure theoretical terms and in computational approaches.

Definitions and notation

We denote by $\chi(G)$, the chromatic number of G, that is the least number of colors in a proper coloring of G. $\omega(G)$ is used to denote the clique number of G, the size of the largest clique in G. $\alpha(G)$ denotes the independence number of G, the size of the largest independent set in G.

We will use confound identity with isomorphism and write $H = G$ for $H \cong G$, and we will use $H \leq G$ and $G \geq H$ to denote that H is a subgraph of G up to isomorphism. A such, $H < G$ and $G > H$ denote that H is a proper subset of G.

An η-star is the closed neighbourhood of a vertex whose (open) neighbourhood is an independent set of size η. A claw is a 3-star, and a 2-star is the same P_2, a path of length 2.

G is H-free if H cannot be found as an induced subgraph of G.

Unless otherwise mentioned, throughout this document G is a k-chromatic double-critical graph.

1 Introduction

A double-critical graph of chromatic number k is a k-chromatic graph that is connected, and such that the deletion of any two adjacent vertices x, y results in a graph $G - \{x, y\}$ that is $(k-2)$-chromatic. In 1967, Erdős conjectured that K_k is the only k-chromatic double critical graph. The conjecture has since remained open with few results known about it.

Conjecture 1.1 (Double-critical graph conjecture)**.** *If G is k-chromatic double-critical, then $G = K_k$.*

1.1 Erdős-Lovász Tihany conjecture

The double-critical graph conjecture is a direct corollary of the following conjecture, also by Erdős and Lovász and of which even less is known:

Conjecture 1.2 (Erdős-Lovász Tihany). *For all $x, t \geq 2$, and all graphs G, if $\omega(G) < \chi(G)$, then there exist disjoint $G_1, G_2 \leq G$ such that $\chi(G_1) \geq s$ and $\chi(G_2) \geq t$.*

The double critical graph conjecture 1.1 follows from the Erdős-Lovász Tihany conjecture 1.2 in the case $s = 2$ and with (possibly) further assumptions.

Proof. Let G be a k-chromatic double critical graph. Take $s = 2, t = k - 1$. We want to proceed with contraposition. For any $G_1 \leq G$, if $\chi(G_1) < s = 2$, then we have $\omega(G) = \chi(G)$. Otherwise, if $\chi(G_1) \geq 2$, then, G_1 has an edge xy. Since $G_2 \leq G$ and $G_1 \cap G_2 = \emptyset$, $G_2 \leq G - \{x, y\}$, and G is double-critical, so $\chi(G_2) \leq k - 2 = t - 1$ and again we have $\omega(G) = \chi(G)$. Take a $K_k \leq G$. Select any edge uv of G, delete u, v. You have that $G - \{u, v\}$ is $k - 2$ chromatic, which means that $u, v \in K_k \leq G$. Finally, if w has no edge on it, then it is an isolated vertex, which cannot happen since G is connected and with $v(G) \geq 1$. □

2 Theory

2.1 η-stars

The conjecture 1.1 can be studied easier on graphs which are η-star-free. The reason for that is illustrated in the following sequence of theorems.

Lemma 2.1. *For any η-star-free graph G, if $v \in v(G)$, $\alpha(G[N(v)]) = \eta - 1$.*

Proof. Suppose $S \in G[N(v)]$ is an independent set with at least η vertices, then, $G[S + v] \leq G$ is an induced η-star. □

Theorem 2.2. *If G is k-chromatic double-critical and 2-star-free, then $G = K_k$.*

Proof. G is connected. Take any $x, y \in v(G)$, and an xy-path in G. By induction, assume you have a path of length 2, since a 2-star is also simply a P_2, we must instead have a triangle, so that $xy \in e(G)$. So $G = K_k$. □

Theorem 2.3. *If G is 5-chromatic double-critical and 3-star free, then $G = K_5$.*

Proof. Take x, y adjacent in G; then, by 2.6 they must share $5-2 = 3$ neighbours u_1, u_2, u_3. But by lemma 2.1, there must be an within $\{u_1, u_2, u_3\}$. Without loss of generality, assume $u_1u_2 \in e(G)$. Then, $\{x, y, u_1, u_2\} = K_4 \leq G$; but then by 2.5, we have $G = K_5$. □

Theorem 2.4 (Rolek, Song 2017 [RS17b]). *If G is k-chromatic double critical and 3-star free, and $k \leq 8$, then $G = K_k$.*

Theorem 2.5 (Huang, Yu 2016 [HY16]). *If $K_{k-1} \leq G$, then $G = K_k$.*

Theorem 2.6 (Huang, Yu 2016 [HY16]). *If $xy \in E(G)$, then x, y share at least one common neighbour in each of the $k - 2$ color-sets of $G - \{x, y\}$.*

These theorems show how forbidding stars allows one to add edges enough to get complete graphs. Lemma 2.1 captures the essence of the power of η-star-free conditions and can easily prove something like the following.

Theorem 2.7. *If G is a k-chromatic double-critical graph that is also η-star-free with $\eta \leq k-2$, and $v \in V(G)$, then $G[N(v)]$ has at least $\frac{1}{2-\log_2 3}(k - l + 1)$ edges.*

Proof. We have 2^n subsets of the vertices of $G[N(v)]$. If we add an edge xy, then at worst we could make $\frac{1}{4}$ of the sets become non-independent. We also know that we have at most $2^{\eta-1}$ independent sets. By induction, if we add p edges, we have at least $2^n \cdot \left(\frac{3}{4}\right)^p$ independent sets remaining. Therefore, $2^n \cdot \left(\frac{3}{4}\right)^p \leq 2^{\eta-1}$. From that we obtain $p \geq \frac{1}{2-\log_2 3}(k - l + 1)$. □

2.2 Minors

Interesting work has been also done by Rolek and Song in 2017 [RS17a] on a weakening of the conjecture by Kawarabayashi et al.

Conjecture 2.8 (Kawarabayashi, Pedersen, Toft, 2010 [KPT10]). *Every k-chromatic double-critical graph has a K_k minor.*

As far as our research has reached, the latest work on minors is by Rolek and Song in 2017 [RS17a] who proved 2.8 until $k \leq 9$. In contrast, the highest number reached for claw-free graphs is $k \leq 8$ also by Rolek and Song in 2017 [RS17b], and the highest number reached for general graphs remains $k \leq 5$ proved in 1986 by Stiebitz with the note that $k = 2, 3, 4$ are trivial or almost trivial.

3 Computational approaches

3.1 Latest results

Although computational approaches cannot reach positive answers, there has been some work on searching for counter-examples, so far unproductively. Computational approaches to this problem are riddled with issues and difficulties.

To the best of our knowledge, the latest results computationally were reached 8 years ago in 2012 by Pedersen who was not able to disprove the conjecture for graphs with less than 13 vertices.

Theorem 3.1 (Pedersen, 2012 [Ped12])**.** *There is no non-complete graph on less than 13 vertices.*

3.2 Issues

There are many obstacles against computational searches for this problem which we will now briefly highlight.

Infinite search space for fixed k

First, it is impossible to settle the issue for any fixed chromatic number computationally, because a fixed chromatic number k does not enforce any upper bound on the number of vertices of G.

Coloring is NP-complete

Almost any approach to computationally solve the issue requires finding chromatic numbers of a graph – a problem famously known to be NP-complete.

Search space is too big

Even for a fixed number n of vertices, the naive search space of labeled graphs has size $2^{\binom{n}{2}}$. For $n \geq 13$, this is unreasonable.

3.3 Requirements of a practical search program

In order to effectively search for counter-examples, a search program must satisfy certain constraints. We provide here a list of such constraints with a brief summary of the reasoning behind them:

1. **Resumability**: It ought to be possible to halt the program and resume it at will thus allowing the program to be easily moved between machines and stopped and resumed according to funding and interest. This implies a clear, non-arbitrary method of listing graphs as well as efficient ways to store state and/or results.

2. **High parallelisability**: With high parallelisability, we get access to modern multi-core machines, as well as the possibility of using cluster computing technology (e.g. BOINC) which would allow slow, but sustained computation for low cost. This also implies that the computations should be as **cross-platform** as possible, not relying on specific architectures.

3. **Configurability**: It is essential that one implements a modular program that can run different algorithms on different classes of graphs depending on the current latest literature. For instance, at the moment, there is progress with respect to claw-free graphs and it might be more efficient and useful to restrict the search to graphs with claws as searching for claws is polynomial and fast, while coloring is NP-complete. Later, according to theoretical progress (e.g. general η-stars), that might change, and the program must be able to adjust.

References

[HY16] Hao Huang and Alexander Yu. A note on the double-critical graph conjecture. *arXiv preprint arXiv:1604.05262*, 2016.

[KPT10] Ken-Ichi Kawarabayashi, Anders Sune Pedersen, and Bjarne Toft. Double-critical graphs and complete minors. *The Electronic Journal of Combinatorics*, 17(1), 2010.

[Ped12] Anders Sune Pedersen. No non-complete double-critical graph on less than 13 vertices. 2012.

[RS17a] Martin Rolek and Zi-Xia Song. Clique minors in double-critical graphs. *Journal of Graph Theory*, 88(2):347–355, 2017.

[RS17b] Martin Rolek and Zi-Xia Song. Double-critical graph conjecture for claw-free graphs. *Discrete Mathematics*, 340(7):1633–1638, 2017.